Kuwait University
Dept.of Math.&Comp.Sci.

Math.102
Midterm 2

May 6, 2004
Duration: 75 min

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1. (2 pts. each) Find the limit, if exists.

(a) $\lim_{x\to\left(\frac{\pi}{2}\right)^+} (\sec x - \tan x)$

(b) $\lim_{x\to 0^+} (1-3x)^{\cot x}$

2. (3 pts. each) Evaluate the following integrals

(a) $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

(b) $\int \frac{x^2+x+1}{(x^2+2)(x-1)}\,dx$

(c) $\int \frac{\csc^3 x \cot^3 x}{\sqrt{\sin x}}\,dx$

(d) $\int \frac{dx}{3+\cos x - 2\sin x}$

(e) $\int \frac{x\tan^{-1} x}{\sqrt{1+x^2}}\,dx$

3. (3 pts. each) Check the improper integral for convergence, and find its value if it is convergent.

(a) $\int_{-\infty}^{1} \frac{dx}{x^2+2x+5}$

(b) $\int_0^4 (2-x)^{-\frac{4}{3}}\,dx$

Total: 25 points

Calculators and mobile phones are not allowed.

| Second Midterm |

(a) $\sec x - \tan x = \dfrac{1-\sin x}{\cos x}$

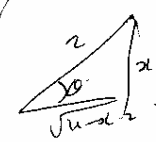
$$\lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)'}{(\cos x)'} = \lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{-\sin x} = 0 \Rightarrow \lim_{x\to\frac{\pi}{2}} (\sec x - \tan x) = 0$$

(b) $\dfrac{\cos x \ln(1-3x)}{\sin x} \sim \dfrac{-\sin x \ln(1-3x) + \frac{\cos x}{1-3x}(-3)}{\cos(x)} \xrightarrow[x\to 0]{} -3$

$\dfrac{\ln(1-3x)}{\tan x} \overset{L'H}{=} \dfrac{\frac{-3}{1-3x}}{\sec^2 x}\Big|_{x=0} = \dfrac{\frac{-3}{1-0}}{1} = -3$

$$\Rightarrow \lim_{x\to 0+} (1-3x)^{\cot x} = e$$

$\sin\theta = \frac{x}{2}$.



(a) $x = 2\sin t$, $dx = 2\cos t\, dt$, $\sqrt{4-x^2} = 2\cos t$

$$I = \int \frac{4\sin^2 t}{2\cos t} 2\cos t\, dt = 4\int \sin^2 t\, dt = 2\int (1-\cos 2t)\, dt$$

$$= 2t - \sin 2t = 2\left(\sin^{-1}\frac{x}{2} - \frac{1}{4} \times \sqrt{4-x^2}\right) + C$$

(b) $\dfrac{x^2+x+1}{(x^2+2)(x-1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+2}$

$A(x^2+2) + (Bx+C)(x-1) = x^2+x+1$

$x = 1 \Rightarrow 3A = 3 \Rightarrow A = 1$

$x = 0 \Rightarrow 2 - C = 1 \Rightarrow C = 1$

$x = -1 \Rightarrow 3 + (-B+1)(-2) = 1 \Rightarrow B = 0$

$$\Rightarrow I = \ln|x-1| + \frac{1}{\sqrt{2}} \tan^{-1}\frac{x}{\sqrt{2}} + C$$

(c) $I = \int \csc^{\frac{5}{2}} x \cot^3 x\, dx = -\int u^{\frac{3}{2}}(u^2-1)\, du = \int (-u^{\frac{7}{2}} + u^{\frac{3}{2}})\, du$

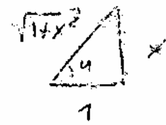
$u = \csc x$, $du = -\csc x \cot x$, $\cot^2 x = u^2 - 1$

$I = -\frac{2}{9} \csc^{9/2} x + \frac{2}{5} \csc^{5/2} x + C$

(d) $u = \tan\frac{x}{2}$, $3 + \cos x - 2\sin x = \dfrac{3+3u^2+1-u^2-4u}{1+u^2} = \dfrac{2((u-1)^2+1)}{1+u^2}$

$$I = \int \frac{du}{(u-1)^2+1} = \tan^{-1}\left(\tan\frac{x}{2} - 1\right) + C$$

(e) $\quad I = \sqrt{1+x^2}\tan^{-1}x - \int \frac{dx}{\sqrt{1+x^2}}$



$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2 u}{\sec u}\,du = \ln|\sec u + \tan u| + C$$

$x = \tan u \qquad = \ln\left|\sqrt{1+x^2} + x\right| + C$

or

$x = \sinh u$

$$\int \frac{\cosh u}{\cosh u}\,du = u + C = \sinh^{-1}x + C$$

$$\Rightarrow I = \sqrt{1+x^2}\tan^{-1}x - \ln\left|\sqrt{1+x^2} + x\right| + C$$

3. (a) $\quad \int \frac{dx}{(x^2+1)+4} = \frac{1}{2}\tan^{-1}\frac{x+1}{2} + C$

$$I = \frac{1}{2}\lim_{t\to-\infty}\left(\tan^{-1}1 - \tan^{-1}\frac{t+1}{2}\right) = \frac{1}{2}\left(\frac{\pi}{4} + \frac{\pi}{2}\right) = \frac{3\pi}{8}$$

(b) Discontinuity at $x = 2$

$$\int_0^t (2-x)^{-\frac{4}{3}}dx = 3(2-x)^{-\frac{1}{3}}\Big|_0^t = 3\left((2-t)^{-\frac{1}{3}} - 2^{-\frac{1}{3}}\right) \xrightarrow[t\to 2^-]{} \infty$$

$\Rightarrow$ Integral is divergent